命题逻辑系统 $R_0Ł_{3n+1}$ 中公式的 Γ-真度及性质

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摘 要 计量逻辑理论是王国俊教授于21世纪初期建立的一种新型逻辑理论,真度理论在计量逻辑理论中发挥着关键的作用.该文将真度概念加以推广,在$(3n+1)$-值模糊命题逻辑系统$R_0Ł$中引入了公式相对于含有限个命题变元的理论的Γ-真度;讨论了与析取连接词、合取连接词、蕴含连接词、否定连接词等基本逻辑连接词相关的Γ-真度性质;讨论了与分离规则MP,三段论规则HS等推理规则相关的Γ-真度性质.该文的工作为将计量逻辑的思想融入$(3n+1)$-值模糊命题逻辑系统$R_0Ł$并建立基于给定理论的近似推理基本框架和相关的逻辑度量空间奠定了基础.

关键词 模糊逻辑;计量逻辑;命题逻辑系统$R_0Ł$;Γ-真度;连接词;MP规则;HS规则

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The Γ-Truth Degrees of Formulas in Propositional Logic System $R_0Ł_{3n+1}$ with Properties

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Abstract The theory of quantitative logic is a new theory of logic established by Wang Guojun in the early part of 21st century, and theory of truth degrees of formulas plays a key role in the theory of quantitative logic. In this paper, the concept of truth degree of a formula is generalized to Γ-truth degree of a formula relative to the theory Γ which contains finite propositional variables in $(3n+1)$-valued propositional logic system $R_0Ł$; the basic properties of Γ-truth degrees relative to connectives of disjunction, conjunction, implication, negation are investigated; the basic properties of Γ-truth degrees relative to inference rules of Modus ponens, Hypothetical syllogism are discussed. The work presented in this paper is a basis for introducing the idea of quantitative logic to $(3n+1)$-valued propositional logic system $R_0Ł$ and establishing framework of approximate reasoning based on a given theory and relevant logic metric spaces.

Keywords fuzzy logic; quantitative logic; propositional logic system $R_0Ł$; Γ-truth degree; connectives; modus ponens; hypothetical syllogism

1 引 言

20世纪70年代王国俊和Pavelka[1-2]在格值逻辑框架下从语义和语构两方面提出了基于理论Γ的逻辑结论的程度化理论,将数值计算的精确性和形式逻辑的严谨性有机地结合在一起.21世纪初期王国俊[3-4]基于均匀概率的思想建立的计量逻辑理论是逻辑结论程度化方法的又一具体表现.目前计量逻辑理论已经成功应用于几种重要的命题逻辑系

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统中，并且取得了一系列重要成果[5-17]. 文献[8]在经典二值逻辑命题逻辑系统中对真度理论进行了研究；文献[9]讨论了 n-值 Łukasiewicz 命题逻辑系统中的真度性质；文献[13]在 n-值 R_0-型命题逻辑系统中对广义真度进行了研究；文献[15]将计量逻辑的思想引入多值模态逻辑中；文献[14]采用公理化方法在一类一阶逻辑公式中研究了真度理论及其应用；文献[10-12]在几种标准完备逻辑系统中利用真度理论建立了相容理论的程度化方法和升级推理方法.

2001 年，Esteva 和 Gödo[18] 基于左连续(非连续)t-模建立了一种新型命题逻辑系统 NM. 2005 年，王三民[19-20] 综合 Łukasiewicz 命题逻辑系统和 NM 命题逻辑系统中的重言式建立了 NMŁ 命题逻辑系统和相应的 NMŁ 逻辑代数[19-20]. 本文的目的是借助文献[13]的方法，在有限值命题逻辑系统 NMŁ 中建立公式相对于理论 Γ 的真度理论. 由于 NMŁ 逻辑代数定义的特殊性，本文在$(3n+1)$-值命题逻辑系统 NMŁ 中建立了公式相对于理论 Γ 的真度理论，而对于在 $3n$-值和 $3n+2$-值命题逻辑系统 NMŁ 中建立公式相对于理论 Γ 的真度理论的方法似乎并不能从本文的方法类比得到.

本文首先在$(3n+1)$-值命题逻辑系统 NMŁ 中提出了公式诱导函数的概念. 其次，借助于诱导函数定义了公式相对于含有限个命题变元的理论的 Γ-真度，并对其基本性质进行了讨论. 随后，分别给出了与析取连接词、合取连接词、蕴含连接词、否定连接词等基本连接词相关的 Γ-真度规则，讨论了和推理规则，三段论规则相关的 Γ-真度性质. 从而为在$(3n+1)$-值命题逻辑系统 NMŁ 中展开 Γ-近似推理，为将计量逻辑的思想方法融入有限值命题逻辑系统 NMŁ 奠定了必要基础.

由于 NM 命题逻辑系统和 R_0-型命题逻辑系统是相互等价的命题逻辑系统[21]，本文将 NMŁ 命题逻辑系统记为 $R_0Ł$ 命题逻辑系统，将 NMŁ 代数记为 $R_0Ł$ 代数. $(3n+1)$-值命题逻辑系统 $R_0Ł$，$(3n+1)$ 值 $R_0Ł$ 代数等基本定义请参见文献[7]，其他未定义的概念和符号请参见文献[1,19].

2　$R_0Ł_{3n+1}$ 系统中公式的 Γ-真度

本节首先给出命题逻辑系统 $R_0Ł_{3n+1}$ 中 Γ-真度的定义，其次讨论其基本性质.

定义 1.　在命题逻辑系统 $R_0Ł_{3n+1}$ 中，设 $\varphi=\varphi(p_{i_1},\cdots,p_{i_m})$ 是公式集 $\mathfrak{I}$ 中含有 m 个命题变元的公式，在 φ 中将 p_{i_j} 换为 $x_{i_j}(j=1,\cdots,m)$，将 $\bar{0}$ 换为 0，则 φ 可诱导映射 $\bar{\varphi}:R_0Ł_{3n+1}^m\to R_0Ł_{3n+1}$，其定义为 $\bar{\varphi}(x_{i_1},\cdots,x_{i_m})$ 由 $x_{i_1},\cdots,x_{i_m}$ 通过“$*$”，“$\Rightarrow$”，“$\vee$”和“$\wedge$”连接而成，其方式恰如 φ 由 $p_{i_1},\cdots,p_{i_m}$ 通过逻辑连接词“&”，“$\to$”，“$\vee$”和“$\wedge$”连接而成的方式. 称 $\bar{\varphi}$ 为 φ 在 $R_0Ł_{3n+1}$-代数上诱导的函数.

定义 2.　设 $f:R_0Ł_{3n+1}^k\to R_0Ł_{3n+1}$，定义 $f^{(m)}:R_0Ł_{3n+1}^m\to R_0Ł_{3n+1}$，$\forall(x_{j_1},\cdots,x_{j_k},\cdots,x_{j_m})\in R_0Ł_{3n+1}^m$，$f^{(m)}(x_{j_1},\cdots,x_{j_k},\cdots,x_{j_m})=f(x_{i_1},\cdots,x_{i_k})$. 称 $f^{(m)}$ 为 f 的直到 m 次扩张. 这里 $k\le m$，$\{i_1,\cdots,i_k\}\subseteq\{j_1,\cdots,j_k,\cdots,j_m\}$.

定义 3.　在 $R_0Ł_{3n+1}$ 系统中，设 $\Gamma\subseteq\mathfrak{I}$，$\varphi\in\mathfrak{I}$，$S(\Gamma)=\{p\in S\mid\exists\chi\in\Gamma,p$ 是构成 χ 的原子命题$\}$有限，$S(\Gamma\cup\{\varphi\})=\{p_1,\cdots,p_m\}$. $\forall i\in\{0,1,\cdots,3n\}$，令 $N(\Gamma,\varphi,i)=\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma)\,\middle|\,\bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\right\}\right|$，$N(\Gamma)=\left\{(x_1,\cdots,x_m)\in R_0Ł_{3n+1}\,\middle|\,\forall\chi\in\Gamma,\bar{\chi}^{(m)}(x_1,\cdots,x_m)=1\right\}$. 定义：

$$\tau_\Gamma(\varphi)=\begin{cases}1, & N(\Gamma)=\varnothing\\ \dfrac{1}{|N(\Gamma)|}\sum\limits_{i=0}^{3n}\dfrac{i}{3n}N(\Gamma,\varphi,i), & N(\Gamma)\ne\varnothing\end{cases}.$$

称 $\tau_\Gamma(\varphi)$ 为 φ 在 $R_0Ł_{3n+1}$ 系统中相对于理论 Γ 的 Γ-真度.

定理 1(Γ-真度不变定理).　在 $R_0Ł_{3n+1}$ 系统中，设 $\Gamma\subseteq\mathfrak{I}$，$\varphi\in\mathfrak{I}$，$S(\Gamma\cup\{\varphi\})=\{p_1,\cdots,p_m\}$，$S^*=\{p_1,\cdots,p_m,p_{m+1},\cdots,p_{m+k}\}\subseteq S$. 则

$$\tau_\Gamma(\varphi)=\begin{cases}1, & N^*(\Gamma)=\varnothing\\ \dfrac{1}{|N^*(\Gamma)|}\sum\limits_{i=0}^{3n}\dfrac{i}{3n}N^*(\Gamma,\varphi,i), & N^*(\Gamma)\ne\varnothing\end{cases}.$$

这里，$\forall i\in\{0,1,\cdots,3n\}$，$N^*(\Gamma)=\left\{(x_1,\cdots,x_m,\cdots,x_{m+k})\in R_0Ł_{3n+1}^{m+k}\mid\forall\chi\in\Gamma,\bar{\chi}^{(m+k)}(x_1,\cdots,x_{m+k})=1\right\}$，

$N^*(\Gamma,\varphi,i)=\left|\left\{(x_1,\cdots,x_m,\cdots,x_{m+k})\in N^*(\Gamma)\,\middle|\,\bar{\varphi}^{(m+k)}(x_1,\cdots,x_{m+k})=\frac{i}{3n}\right\}\right|$.

证明.　因为 $N^*(\Gamma)=\{(x_1,\cdots,x_m,\cdots,x_{m+k})\in R_0Ł_{3n+1}^{m+k}\mid\forall\chi\in\Gamma,\bar{\chi}^{(m+k)}(x_1,\cdots,x_{m+k})=1\}$，$N(\Gamma)=\{(x_1,\cdots,x_m)\in R_0Ł_{3n+1}^m\mid\forall\chi\in\Gamma,\bar{\chi}^{(m)}(x_1,\cdots,x_m)=1\}$，$\forall(x_1,\cdots,x_{m+k})\in R_0Ł_{3n+1}^{m+k}$，$\bar{\chi}^{(m+k)}(x_1,\cdots,x_{m+k})=\bar{\chi}^{(m)}(x_1,\cdots,x_m)$，

所以，$|N^*(\Gamma)|=|N(\Gamma)|\times(3n+1)^k$，因此

当 $N^*(\Gamma)=\varnothing$ 时，可知 $N(\Gamma)=\varnothing$，所以 $\tau_\Gamma(\varphi)=1$.

当 $N^*(\Gamma)\neq\varnothing$ 时，可知 $N(\Gamma)\neq\varnothing$，

所以，$\tau_\Gamma(\varphi)=\dfrac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\dfrac{i}{3n}N(\Gamma,\varphi,i)$.

因为 $N^*(\Gamma,\varphi,i)=\left|\left\{(x_1,\cdots,x_m,\cdots,x_{m+k})\in N^*(\Gamma)\,\middle|\,\bar{\varphi}^{(m+k)}(x_1,\cdots,x_{m+k})=\dfrac{i}{3n}\right\}\right|=\left|\left\{(x_1,\cdots,x_m,\cdots,x_{m+k})\in N_\Gamma^*\,\middle|\,\bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\dfrac{i}{3n}\right\}\right|=N(\Gamma,\varphi,i)\times(3n+1)^k$.

因此，

$$\frac{1}{|N^*(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}N^*(\Gamma,\varphi,i)=$$

$$\frac{1}{(3n+1)^k\times|N(\Gamma)|}\times\sum_{i=0}^{3n}\frac{i}{3n}(\Gamma,\varphi,i)\times(3n+1)^k$$

$$=\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}N(\Gamma,\varphi,i).$$

所以，

$$\tau_\Gamma(\varphi)=\begin{cases}1, & N^*(\Gamma)=\varnothing\\ \dfrac{1}{|N^*(\Gamma)|}\sum_{i=0}^{3n}\dfrac{i}{3n}N^*(\Gamma,\varphi,i), & N^*(\Gamma)\neq\varnothing\end{cases}.$$

证毕.

为了表述方便，在本文以后的讨论中，将 $N^*(\Gamma)$ 与 $N^*(\Gamma,\varphi,i)$ 仍记作 $N(\Gamma)$ 与 $N(\Gamma,\varphi,i)$.

定理 2（Γ-重言式的 Γ-真度）. 在 $R_0Ł_{3n+1}$ 系统中，设 $\Gamma_1\subseteq\Gamma_2\subseteq\mathfrak{J}$，$S(\Gamma_2)$ 有限，$\varphi\in\mathfrak{J}$. 则

(1) 若 $\Gamma_1\models\varphi$，则 $\tau_{\Gamma_1}(\varphi)=1$；

(2) 若 $\tau_{\Gamma_1}(\varphi)=1$，则 $\tau_{\Gamma_2}(\varphi)=1$.

证明. 设 $S(\Gamma_2\cup\{\varphi\})=\{p_1,\cdots,p_m\}$，于是

$$N(\Gamma_1)=\{(x_1,\cdots,x_m)\in R_0Ł_{3n+1}^m\mid \forall\chi\in\Gamma_1,\bar{\chi}^{(m)}(x_1,\cdots,x_m)=1\},$$

$$N(\Gamma_2)=\{(x_1,\cdots,x_m)\in R_0Ł_{3n+1}^m\mid \forall\chi\in\Gamma_2,\bar{\chi}^{(m)}(x_1,\cdots,x_m)=1\},$$

$$N(\Gamma_1,\varphi,i)=\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma_1)\,\middle|\,\bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\right\}\right|,$$

$$N(\Gamma_2,\varphi,i)=\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma_2)\,\middle|\,\bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\right\}\right|.$$

1) 当 $N(\Gamma_1)=\varnothing$ 时，可知 $\tau_{\Gamma_1}(\varphi)=1$.

当 $N(\Gamma_1)\neq\varnothing$ 时，由于 $\Gamma_1\models\varphi$，即

$\forall(x_1,\cdots,x_m)\in N(\Gamma_1)$，$\bar{\varphi}^{(m)}(x_1,\cdots,x_m)=1$，

所以，当 $i=0,\cdots,3n-1$ 时，$N(\Gamma_1,\varphi,i)=0$，

而且 $N(\Gamma_1,\varphi,3n)=|N(\Gamma_1)|$.

因此，$\tau_{\Gamma_1}(\varphi)=\dfrac{1}{|N(\Gamma_1)|}\sum_{i=0}^{3n}\dfrac{i}{3n}N(\Gamma_1,\varphi,i)=1$.

2) 由于 $\Gamma_1\subseteq\Gamma_2$，所以 $N(\Gamma_2)\subseteq N(\Gamma_1)$.

当 $N(\Gamma_2)=\varnothing$ 时，可知 $\tau_{\Gamma_2}(\varphi)=1$.

当 $N(\Gamma_2)\neq\varnothing$ 时，可知 $N(\Gamma_1)\neq\varnothing$. 因为

$\tau_{\Gamma_1}(\varphi)=1$，所以 $\dfrac{1}{|N(\Gamma_1)|}\sum_{i=0}^{3n}\dfrac{i}{3n}N(\Gamma_1,\varphi,i)=1$，

因此，$N(\Gamma_1,\varphi,3n)=N(\Gamma_1)$，即

$\forall(x_1,\cdots,x_m)\in N(\Gamma_1)$，$\bar{\varphi}^{(m)}(x_1,\cdots,x_m)=1$，

所以，$\forall(x_1,\cdots,x_m)\in N(\Gamma_2)$，$\bar{\varphi}^{(m)}(x_1,\cdots,x_m)=1$.

因此当 $i=0,\cdots,3n-1$ 时，$N(\Gamma_2,\varphi,i)=0$，

而且 $N(\Gamma_2,\varphi,3n)=|N(\Gamma_2)|$. 所以，

$$\tau_{\Gamma_2}(\varphi)=\frac{1}{|N(\Gamma_2)|}\times\sum_{i=0}^{3n}\frac{i}{3n}N(\Gamma_2,\varphi,i)=1.$$

证毕.

3 Γ-真度连接词规则

本节讨论命题逻辑系统 $R_0Ł_{3n+1}$ 中连接词的 Γ-真度规则，它们在计量逻辑的性质证明中发挥着重要作用[3-4].

引理 1（Γ-真度蕴涵规则）. 在 $R_0Ł_{3n+1}$ 系统中，设 $\Gamma\subseteq\mathfrak{J}$，$S(\Gamma)$ 有限，$\varphi,\psi\in\mathfrak{J}$. 若 $\Gamma_1\models\varphi\to\psi$，则 $\tau_\Gamma(\varphi)\leq\tau_\Gamma(\psi)$.

证明. 见附录 1 引理 1 的证明.

推论 1. 在 $R_0Ł_{3n+1}$ 系统中，设 $\Gamma\subseteq\mathfrak{J}$，$S(\Gamma)$ 有限，$\varphi,\psi\in\mathfrak{J}$. 若 $\Gamma\models\varphi\to\psi$，且 $\Gamma\models\psi\to\varphi$，则 $\tau_\Gamma(\varphi)=\tau_\Gamma(\psi)$.

证明. 引理 1 的直接结论.

定理 3（Γ-真度合取，析取规则）. 在 $R_0Ł_{3n+1}$ 系统中，设 $\Gamma\subseteq\mathfrak{J}$，$S(\Gamma)$ 有限，$\varphi,\psi\in\mathfrak{J}$，则

$$\tau_\Gamma(\varphi\vee\psi)=\tau_\Gamma(\varphi)+\tau_\Gamma(\psi)-\tau_\Gamma(\varphi\wedge\psi).$$

证明. 设 $S(\Gamma\cup\{\varphi,\psi\})=\{p_1,\cdots,p_m\}$，于是

$$N(\Gamma)=\{(x_1,\cdots,x_m)\in R_0Ł_{3n+1}^m\mid \forall\chi\in\Gamma,\bar{\chi}^{(m)}(x_1,\cdots,x_m)=1\}.$$

当 $N(\Gamma)=\varnothing$ 时，由于

$$\tau_\Gamma(\varphi\vee\psi)=\tau_\Gamma(\varphi)=\tau_\Gamma(\psi)=\tau_\Gamma(\varphi\wedge\psi)=1,$$

所以，$\tau_\Gamma(\varphi\vee\psi)=\tau_\Gamma(\varphi)+\tau_\Gamma(\psi)-\tau_\Gamma(\varphi\wedge\psi)$.

当 $N(\Gamma)\neq\varnothing$ 时，$\forall i\in\{0,1,\cdots,m\}$，

$$N(\Gamma,\varphi\vee\psi,i)=\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma)\,\middle|\,\overline{\varphi\vee\psi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\right\}\right|$$

$$=\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma)\,\middle|\,\bar{\varphi}^{(m)}(x_1,\cdots,x_m)\vee\right.\right.$$

$$\bar{\psi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\}\Big|$$

$$=\Big|\{(x_1,\cdots,x_m)\in N(\Gamma)\mid \bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\}\Big|+\Big|\{(x_1,\cdots,x_m)\in N(\Gamma)\mid \bar{\psi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\}\Big|-\Big|\{(x_1,\cdots,x_m)\in N(\Gamma)\mid \bar{\varphi}^{(m)}(x_1,\cdots,x_m)\wedge\bar{\psi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\}\Big|$$

$$=N(\Gamma,\varphi,i)+N(\Gamma,\psi,i)-N(\Gamma,\varphi\wedge\psi,i).$$

所以，

$$\tau_\Gamma(\varphi\vee\psi)=\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}N(\Gamma,\varphi\vee\psi,i)=\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}(N(\Gamma,\varphi,i)+N(\Gamma,\psi,i)-N(\Gamma,\varphi\wedge\psi,i))=\tau_\Gamma(\varphi)+\tau_\Gamma(\psi)-\tau_\Gamma(\varphi\wedge\psi).$$

证毕.

定理 4(Γ-真度否定规则). 在 $R_0Ł_{3n+1}$ 系统中，设 $\Gamma\cup\{\varphi\}\subseteq\Im$，$S(\Gamma)$ 有限. 若 $N(\Gamma)\neq\varnothing$，则

$$\tau_\Gamma(\neg\varphi)=1-\tau_\Gamma(\varphi).$$

证明. 设 $S(\Gamma\cup\{\varphi\})=\{p_1,\cdots,p_m\}$，于是

$$N(\Gamma)=\{(x_1,\cdots,x_m)\in R_0Ł_{3n+1}^m\mid \forall\chi\in\Gamma,\bar{\chi}^{(m)}(x_1,\cdots,x_m)=1\}.$$

因为 $N(\Gamma)\neq\varnothing$，所以，

$$\tau_\Gamma(\neg\varphi)=\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}N(\Gamma,\neg\varphi,i)$$

$$=\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}\Big|\{(x_1,\cdots,x_m)\in N(\Gamma)\mid \overline{\neg\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\}\Big|$$

$$=\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}\Big|\{(x_1,\cdots,x_m)\in N(\Gamma)\mid 1-\bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\}\Big|$$

$$=\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}\Big|\{(x_1,\cdots,x_m)\in N(\Gamma)\mid \bar{\varphi}^{(m)}(x_1,\cdots,x_m)=1-\frac{i}{3n}\}\Big|$$

$$=\frac{1}{|N(\Gamma)|}\sum_{j=3n}^{0}\frac{3n-j}{3n}\Big|\{(x_1,\cdots,x_m)\in N(\Gamma)\mid \bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{j}{3n}\}\Big|$$

$$=\frac{1}{|N(\Gamma)|}\Big(\sum_{j=3n}^{0}\Big|\{(x_1,\cdots,x_m)\in N(\Gamma)\mid \bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{j}{3n}\}\Big|-\sum_{j=3n}^{0}\frac{j}{3n}\Big|\{(x_1,\cdots,x_m)\in N(\Gamma)\mid \bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{j}{3n}\}\Big|\Big)$$

$$=\frac{1}{|N(\Gamma)|}\Big(|N(\Gamma)|-\sum_{i=0}^{3n}\frac{i}{3n}\Big|\{(x_1,\cdots,x_m)\in N(\Gamma)\mid \bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\}\Big|\Big)$$

$$=1-\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}\Big|\{(x_1,\cdots,x_m)\in N(\Gamma)\mid \bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\}\Big|$$

$$=1-\tau_\Gamma(\varphi).$$

证毕.

4 Γ-真度 MP 规则和 Γ-真度 HS 规则

本节将给出推理规则 MP，三段论规则 HS 的 Γ-真度性质，在基于命题逻辑系统的逻辑度量空间和近似推理中他们发挥了关键的作用[3-4].

引理 2. 在 $R_0Ł_{3n+1}$ 系统中，设 $\Gamma\subseteq\Im$，$S(\Gamma)$ 有限，$\varphi,\psi\in\Im$. 令 $\Gamma_1=\Gamma\cup\{\psi\to\varphi\}$，则

$$\tau_{\Gamma_1}(\varphi)+\tau_{\Gamma_1}(\varphi\to\psi)\leq 1+\tau_{\Gamma_1}(\psi).$$

证明. 见附录 2 引理 2 的证明.

定理 5(Γ-真度 MP 规则). 在 $R_0Ł_{3n+1}$ 系统中，设 $\Gamma\subseteq\Im$，$S(\Gamma)$ 有限，$\varphi,\psi\in\Im$. 则

$$\tau_\Gamma(\varphi)+\tau_\Gamma(\varphi\to\psi)\leq 1+\tau_\Gamma(\psi).$$

证明. 见附录 3 定理 5 的证明.

引理 3. 在 $R_0Ł_{3n+1}$ 系统中，设 $\Gamma\subseteq\Im$，$S(\Gamma)$ 有限，$\varphi,\psi,\chi\in\Im$. 若 $\models\varphi\to(\psi\to\chi)$，则

$$\tau_\Gamma(\varphi)+\tau_\Gamma(\psi)\leq 1+\tau_\Gamma(\chi).$$

证明. 由于 $\models\varphi\to(\psi\to\chi)$，所以由引理 1 可得 $\tau_\Gamma(\varphi)\leq\tau_\Gamma(\psi\to\chi)$. 因此由定理 5 可得

$$\tau_\Gamma(\varphi)+\tau_\Gamma(\psi)\leq\tau_\Gamma(\psi)+\tau_\Gamma(\psi\to\chi)\leq 1+\tau_\Gamma(\chi).$$

证毕.

定理 6(Γ-真度 HS 规则). 在 $R_0Ł_{3n+1}$ 系统中，设 $\Gamma\subseteq\Im$，$S(\Gamma)$ 有限，$\varphi,\psi,\chi\in\Im$. 则

$$\tau_\Gamma(\varphi\to\psi)+\tau_\Gamma(\psi\to\chi)\leq 1+\tau_\Gamma(\varphi\to\chi).$$

证明. 根据引理 3 和 $\models(\varphi\to\psi)\to((\psi\to\chi)\to(\varphi\to\chi))$ 可证. 证毕.

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附录 1. 引理 1 的证明.

设 $S(\Gamma\cup\{\varphi,\psi\})=\{p_1,\cdots,p_m\}$,于是

$$N(\Gamma)=\{(x_1,\cdots,x_m)\in R_0Ł_{3n+1}^m \mid \forall\chi\in\Gamma,\ \bar{\chi}^{(m)}(x_1,\cdots,x_m)=1\}.$$

当 $N(\Gamma)=\varnothing$ 时,命题成立.

当 $N(\Gamma)\neq\varnothing$ 时,由于 $\Gamma\models\varphi\to\psi$,即

$$\forall(x_1,\cdots,x_m)\in N(\Gamma),\ \overline{\varphi\to\psi}^{(m)}(x_1,\cdots,x_m)=1,$$

所以,$\bar{\varphi}^{(m)}(x_1,\cdots,x_m)\le\bar{\psi}^{(m)}(x_1,\cdots,x_m)$.

因此,$\forall i\in\{0,1,\cdots,3n\}$,

$$\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma)\,\middle|\,\bar{\varphi}^{(m)}(x_1,\cdots,x_m)\ge\frac{i}{3n}\right\}\right|\le$$

$$\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma)\,\middle|\,\bar{\psi}^{(m)}(x_1,\cdots,x_m)\ge\frac{i}{3n}\right\}\right|,$$

即 $\sum_{j=i}^{3n}N(\Gamma,\varphi,i)\le\sum_{j=i}^{3n}N(\Gamma,\varphi,i)$,

因此，

$$\sum_{i=0}^{3n}\frac{i}{3n}N(\Gamma,\varphi,i)=\frac{1}{3n}N(\Gamma,\varphi,1)+\frac{2}{3n}N(\Gamma,\varphi,2)+\cdots+\frac{3n-1}{3n}N(\Gamma,\varphi,3n-1)+N(\Gamma,\varphi,3n)$$

$$=\frac{1}{3n}N(\Gamma,\varphi,1)+\cdots+\frac{3n-1}{3n}N(\Gamma,\varphi,3n-1)-(N(\Gamma,\psi,3n)-N(\Gamma,\varphi,3n))+N(\Gamma,\psi,3n)$$

$$\leq\frac{1}{3n}N(\Gamma,\varphi,1)+\cdots+\frac{3n-1}{3n}N(\Gamma,\varphi,3n-1)-\frac{3n-1}{3n}(N(\Gamma,\psi,3n)-N(\Gamma,\varphi,3n))+N(\Gamma,\psi,3n)$$

$$=\frac{1}{3n}N(\Gamma,\varphi,1)+\cdots+\frac{3n-2}{3n}N(\Gamma,\varphi,3n-2)-\frac{3n-1}{3n}((N(\Gamma,\psi,3n-1)+N(\Gamma,\psi,3n))-(N(\Gamma,\varphi,3n-1)+N(\Gamma,\varphi,3n)))+\left(\frac{3n-1}{3n}N(\Gamma,\psi,3n-1)+N(\Gamma,\psi,3n)\right)$$

$$\leq\frac{1}{3n}N(\Gamma,\varphi,1)+\cdots+\frac{3n-2}{3n}N(\Gamma,\varphi,3n-2)-\frac{3n-2}{3n}\Big(\sum_{j=3n-1}^{3n}N(\Gamma,\psi,j)-\sum_{j=3n-1}^{3n}N(\Gamma,\varphi,j)\Big)+\sum_{j=3n-1}^{3n}\frac{j}{3n}N(\Gamma,\psi,j)$$

$$=\cdots\cdots$$

$$\leq-\frac{1}{3n}\Big(\sum_{j=1}^{3n}N(\Gamma,\psi,j)-\sum_{j=1}^{3n}N(\Gamma,\varphi,j)\Big)+\sum_{j=1}^{3n}\frac{j}{3n}N(\Gamma,\psi,j)$$

$$\leq\sum_{j=1}^{3n}\frac{j}{3n}(\Gamma,\psi,j)=\sum_{i=0}^{3n}\frac{i}{3n}(\Gamma,\psi,i).$$

所以，$\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}N(\Gamma,\varphi,i)\leq\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}N(\Gamma,\psi,i)$

即 $\tau_\Gamma(\varphi)\leq\tau_\Gamma(\psi)$.

附录 2.　引理 2 的证明.

设 $S(\Gamma\cup\{\varphi,\psi\})=\{p_1,\cdots,p_m\}$，于是

$$N(\Gamma)=\{(x_1,\cdots,x_m)\in R_0Ł_{3n+1}^m\mid\forall\chi\in\Gamma,\bar{\chi}^{(m)}(x_1,\cdots,x_m)=1\},$$

$$N(\Gamma_1)=\{(x_1,\cdots,x_m)\in N(\Gamma)\mid\bar{\psi}^{(m)}(x_1,\cdots,x_m)\leq\bar{\varphi}^{(m)}(x_1,\cdots,x_m)\}.$$

若 $N(\Gamma_1)=\varnothing$，于是公式的真度皆为 1，命题成立.

若 $N(\Gamma_1)\neq\varnothing$，令

$$[i,j]=\left\{(x_1,\cdots,x_m)\in R_0Ł_{3n+1}^m\,\middle|\,\bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n},\bar{\psi}^{(m)}(x_1,\cdots,x_m)=\frac{j}{3n}\right\}.$$

则 $\forall i,j,k,t\in\{0,1,\cdots,3n\}$，当 $(i,j)\neq(k,t)$ 时，$[i,j]\cap[k,t]=\varnothing$.

因此，$|[i,j]\cup[k,t]|=|[i,j]|+|[k,t]|$，

从而，$|N(\Gamma_1)|=\sum_{j=0}^{3n}\sum_{i=j}^{3n}|[i,j]|$，

$$N(\Gamma_1,\varphi,i)=\sum_{j=0}^{i}|[i,j]|,\ N(\Gamma_1,\psi,i)=\sum_{j=i}^{3n}|[j,i]|.$$

当 $1\leq i\leq n$ 时，

$$N(\Gamma_1,\varphi\to\psi,i)=\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma)\,\middle|\,\bar{\psi}^{(m)}(x_1,\cdots,x_m)<\bar{\varphi}^{(m)}(x_1,\cdots,x_m),(1-\bar{\varphi}^{(m)}(x_1,\cdots,x_m))\vee\bar{\psi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\right\}\right|$$

$$=|[3n-i,0]|+|[3n-i,1]|+\cdots+|[3n-i,i]|+\cdots+|[3n,i]|$$

$$=\sum_{j=0}^{i}|[3n-i,j]|+\sum_{j=3n-i+1}^{3n}|[j,i]|,$$

当 $i=n+1$ 时，

$$N(\Gamma_1,\varphi\to\psi,i)=\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma)\,\middle|\,\bar{\psi}^{(m)}(x_1,\cdots,x_m)<\bar{\varphi}^{(m)}(x_1,\cdots,x_m),\ \overline{\varphi\to\psi}^{(m)}(x_1,\cdots,x_m)=\frac{n+1}{3n}\right\}\right|$$

$$=\left|\left\{(x_1,\cdots,x_m)\in N_\Gamma\,\middle|\,\bar{\psi}^{(m)}(x_1,\cdots,x_m)<\bar{\varphi}^{(m)}(x_1,\cdots,x_m),(1-\bar{\varphi}^{(m)}(x_1,\cdots,x_m))\vee\bar{\psi}^{(m)}(x_1,\cdots,x_m)=\frac{n+1}{3n}\right\}\right|$$

$$=|[2n-1,0]|+|[2n-1,1]|+\cdots+|[2n-1,n]|+\cdots+|[3n,n+1]|$$

$$=\sum_{j=0}^{n}|[2n-1,j]|+\sum_{j=2n}^{3n}|[j,n+1]|.$$

当 $n+2\leq i\leq 2n-1$ 时，

$$N(\Gamma_1,\varphi\to\psi,i)=\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma)\,\middle|\,\bar{\psi}^{(m)}(x_1,\cdots,x_m)<\bar{\varphi}^{(m)}(x_1,\cdots,x_m),\ \overline{\varphi\to\psi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\right\}\right|$$

$$=|[2n,i]|+\cdots+|[3n,i]|+|[2n-1,i-1]|+|[2n-2,i-2]|+\cdots+|[2n-i+n+1,n+1]|+|[3n-i,0]|+\cdots+|[3n-i,n]|$$

$$=\sum_{j=2n}^{3n}|[j,i]|+\sum_{j=n+1}^{i-1}|[2n-i+j,j]|+\sum_{j=0}^{n}|[3n-i,j]|$$

当 $2n\leq i<3n$ 时，

$$N(\Gamma_1,\varphi\to\psi,i)=\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma)\,\middle|\,\bar{\psi}^{(m)}(x_1,\cdots,x_m)<\bar{\varphi}^{(m)}(x_1,\cdots,x_m),(1-\bar{\varphi}^{(m)}(x_1,\cdots,x_m))\vee\bar{\psi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\right\}\right|$$

$$=|[3n-i,0]|+|[3n-i,1]|+\cdots+|[3n-i,3n-i-1]|+|[i+1,i]|+|[i+2,i]|+\cdots+|[3n,i]|$$

$$=\sum_{j=0}^{3n-i-1}|[3n-i,j]|+\sum_{j=i+1}^{3n}|[j,i]|.$$

当 $i=3n$ 时，

$$N(\Gamma_1,\varphi\to\psi,3n)=\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma)\;\middle|\;\bar{\psi}^{(m)}(x_1,\cdots,x_m)=\bar{\varphi}^{(m)}(x_1,\cdots,x_m)\right\}\right|=\sum_{j=0}^{3n}|[j,j]|.$$

$$\tau_{\Gamma_1}(\varphi)+\tau_{\Gamma_1}(\varphi\to\psi)-\tau_{\Gamma_1}(\psi)$$

$$=\frac{1}{|N(\Gamma_1)|}\left(\sum_{i=0}^{3n}\frac{i}{3n}N(\Gamma_1,\varphi,i)+\sum_{i=0}^{3n}\frac{i}{3n}N(\Gamma_1,\varphi\to\psi,i)-\sum_{i=0}^{3n}\frac{i}{3n}N(\Gamma_1,\psi,i)\right)$$

$$=\frac{1}{3n|N(\Gamma_1)|}\left[\sum_{i=0}^{3n}i\sum_{j=0}^{i}|[i,j]|+\sum_{i=0}^{3n}iN(\Gamma_1,\varphi\to\psi,i)-\sum_{i=0}^{3n}i\sum_{j=i}^{3n}|[i,j]|\right]$$

$$=\frac{1}{3n|N(\Gamma_1)|}\left[\sum_{i=0}^{3n}i\sum_{j=0}^{i}|[i,j]|+\left(\sum_{i=0}^{n}i\left(\sum_{j=0}^{i}|[3n-2,j]|+\sum_{j=3n-i+1}^{3n}|[j,i]|\right)+(n+1)\left(\sum_{j=0}^{n}|[2n-1,j]|+\sum_{j=2n}^{3n}|[j,n+1]|\right)+\sum_{i=n+2}^{2n-1}i\left(\sum_{j=2n}^{3n}|[j,i]|+\sum_{j=n+1}^{i-1}|[2n-i+j,j]|+\sum_{j=0}^{n}|[3n-i,j]|\right)+\sum_{i=2n}^{3n-1}i\left(\sum_{j=0}^{3n-i-1}|[3n-i,j]|+\sum_{j=i+1}^{3n}|[j,i]|\right)+3n\sum_{j=0}^{3n}|[j,j]|\right)-\sum_{i=0}^{3n}i\sum_{j=i}^{3n}|[j,i]|\right]$$

$$\le\frac{1}{|N(\Gamma_1)|}\left(\sum_{j=0}^{3n}|[j,0]|+\sum_{j=1}^{3n}|[j,1]|+\cdots+\sum_{j=3n}^{3n}|[j,3n]|\right)$$

$$=\frac{1}{|N(\Gamma_1)|}\sum_{i=0}^{3n}\sum_{j=i}^{3n}|[j,i]|$$

$$=\frac{1}{|N(\Gamma_1)|}\times|N(\Gamma_1)|=1.$$

综上可知，$\tau_{\Gamma_1}(\varphi)+\tau_{\Gamma_1}(\varphi\to\psi)\le 1+\tau_{\Gamma_1}(\psi)$.

附录 3. 定理 5 的证明.

设 $S(\Gamma\cup\{\varphi,\psi\})=\{p_1,\cdots,p_m\}$，于是 $N(\Gamma)=\{(x_1,\cdots,x_m)\in R_0\text{Ł}_{3n+1}^{m}\mid\forall\chi\in\Gamma,\bar{\chi}^{(m)}(x_1,\cdots,x_m)=1\}$. 当 $N(\Gamma)=\varnothing$ 时，命题成立.

当 $N(\Gamma)\ne\varnothing$ 时，令 $\Gamma_1=\Gamma\cup\{\varphi\to\psi\}$，

$$\Gamma_2=\Gamma\cup\{\psi\to\varphi\},\ \Gamma_3=\Gamma\cup\{\varphi\to\psi,\psi\to\varphi\}.$$

于是，

$$N(\Gamma_1)=\{(x_1,\cdots,x_m)\in N(\Gamma)\mid\bar{\varphi}^{(m)}(x_1,\cdots,x_m)\le\bar{\psi}^{(m)}(x_1,\cdots,x_m)\},$$

$$N(\Gamma_2)=\{(x_1,\cdots,x_m)\in N(\Gamma)\mid\bar{\psi}^{(m)}(x_1,\cdots,x_m)\le\bar{\varphi}^{(m)}(x_1,\cdots,x_m)\},$$

$$N(\Gamma_3)=\{(x_1,\cdots,x_m)\in N(\Gamma)\mid\bar{\psi}^{(m)}(x_1,\cdots,x_m)=\bar{\varphi}^{(m)}(x_1,\cdots,x_m)\}.$$

所以 $N(\Gamma_1)\cup N(\Gamma_2)=N(\Gamma)$，$N(\Gamma_1)\cap N(\Gamma_2)=N(\Gamma_3)$，因此，$|N(\Gamma)|=|N(\Gamma_1)|+|N(\Gamma_2)|-|N(\Gamma_3)|$.

$\forall i\in\{0,1,\cdots,3n\}$，

$$N(\Gamma,\varphi,i)=\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma)\;\middle|\;\bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\right\}\right|$$

$$=\left|\left\{(x_1,\cdots,x_m)\in N(\Gamma_1)\cup N(\Gamma_2)\;\middle|\;\bar{\varphi}^{(m)}(x_1,\cdots,x_m)=\frac{i}{3n}\right\}\right|$$

$$=N(\Gamma_1,\varphi,i)+N(\Gamma_2,\varphi,i)-N(\Gamma_3,\varphi,i).$$

同理，

$$N(\Gamma,\varphi\to\psi,i)=N(\Gamma_1,\varphi\to\psi,i)+N(\Gamma_2,\varphi\to\psi,i)-N(\Gamma_3,\varphi\to\psi,i)$$

$$N(\Gamma,\psi,i)=N(\Gamma_1,\psi,i)+N(\Gamma_2,\psi,i)-N(\Gamma_3,\psi,i)$$

因此，

$$\tau_\Gamma(\varphi)+\tau_\Gamma(\varphi\to\psi)-\tau_\Gamma(\psi)=\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}N(\Gamma,\varphi,i)+\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}N(\Gamma,\varphi\to\psi,i)-\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}N(\Gamma,\psi,i)$$

$$=\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}(N(\Gamma_1,\varphi,i)+N(\Gamma_2,\varphi,i)-N(\Gamma_3,\varphi,i))+\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}(N(\Gamma_1,\varphi\to\psi,i)+N(\Gamma_2,\varphi\to\psi,i)-N(\Gamma_3,\varphi\to\psi,i))-\frac{1}{|N(\Gamma)|}\sum_{i=0}^{3n}\frac{i}{3n}(N(\Gamma_1,\psi,i)+N(\Gamma_2,\psi,i)-N(\Gamma_3,\psi,i))$$

$$=\left(\frac{|N(\Gamma_1)|}{|N(\Gamma)|}\tau_{\Gamma_1}(\varphi)+\frac{|N(\Gamma_2)|}{|N(\Gamma)|}\tau_{\Gamma_2}(\varphi)-\frac{|N(\Gamma_3)|}{|N(\Gamma)|}\tau_{\Gamma_3}(\varphi)\right)+\left(\frac{|N(\Gamma_1)|}{|N(\Gamma)|}\tau_{\Gamma_1}(\varphi\to\psi)+\frac{|N(\Gamma_2)|}{|N(\Gamma)|}\tau_{\Gamma_2}(\varphi\to\psi)-\frac{|N(\Gamma_3)|}{|N(\Gamma)|}\tau_{\Gamma_3}(\varphi\to\psi)\right)-\left(\frac{|N(\Gamma_1)|}{|N(\Gamma)|}\tau_{\Gamma_1}(\psi)+\frac{|N(\Gamma_2)|}{|N(\Gamma)|}\tau_{\Gamma_2}(\psi)-\frac{|N(\Gamma_3)|}{|N(\Gamma)|}\tau_{\Gamma_3}(\psi)\right)$$

$$=\frac{|N(\Gamma_1)|}{|N(\Gamma)|}(\tau_{\Gamma_1}(\varphi)+\tau_{\Gamma_1}(\varphi\to\psi)-\tau_{\Gamma_1}(\psi))+\frac{|N(\Gamma_2)|}{|N(\Gamma)|}(\tau_{\Gamma_2}(\varphi)+\tau_{\Gamma_2}(\varphi\to\psi)-\tau_{\Gamma_2}(\psi))-\frac{|N(\Gamma_3)|}{|N(\Gamma)|}(\tau_{\Gamma_3}(\varphi)+\tau_{\Gamma_3}(\varphi\to\psi)-\tau_{\Gamma_3}(\psi)).$$

由于 $\Gamma_1=\Gamma\cup\{\varphi\to\psi\}$，因此 $\Gamma_1\vDash\varphi\to\psi$. 根据定理 2 知：$\tau_{\Gamma_1}(\varphi\to\psi)=1$；根据定理 3 知：$\tau_\Gamma(\varphi)\le\tau_\Gamma(\psi)$. 因此：

$$\tau_{\Gamma_1}(\varphi)+\tau_{\Gamma_1}(\varphi\to\psi)-\tau_{\Gamma_1}(\psi)\le 1,$$

由于 $\Gamma_2=\Gamma\cup\{\psi\to\varphi\}$，根据引理 2 可得

$$\tau_{\Gamma_2}(\varphi)+\tau_{\Gamma_2}(\varphi\to\psi)-\tau_{\Gamma_2}(\psi)\le 1.$$

由于 $\Gamma_3\vDash\varphi\to\psi$，所以根据定理 2 得 $\tau_{\Gamma_3}(\varphi\to\psi)=1$；再结合 $\Gamma_3\vDash\psi\to\varphi$，因此根据推论 1 得 $\tau_{\Gamma_3}(\varphi)=\tau_{\Gamma_3}(\psi)$. 因此，

$$\tau_{\Gamma_3}(\varphi)+\tau_{\Gamma_3}(\varphi\to\psi)-\tau_{\Gamma_3}(\psi)=1.$$

所以，$\tau_\Gamma(\varphi)+\tau_\Gamma(\varphi\to\psi)-\tau_\Gamma(\psi)\le\frac{|N(\Gamma_1)|}{|N(\Gamma)|}+\frac{|N(\Gamma_2)|}{|N(\Gamma)|}-\frac{|N(\Gamma_3)|}{|N(\Gamma)|}=\frac{|N(\Gamma_1)|+|N(\Gamma_2)|-|N(\Gamma_3)|}{|N(\Gamma)|}=1.$

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Background

It is an important mission to combine the rigor of logic with the accuracy of calculation in the study of many-valued propositional logic. In the seventies of twentieth century, Pavelka was suggested a theory of logical conclusion degree from aspects of syntax and semantic in the framework of lattice-valued logic that can be found in his series papers On fuzzy logic (I),(II),(III). In the nineties of twentieth century, the work of A prime of probability logic that was written by Adamer and the paper Averaging the truth-value in Łukasiewicz logic that was written by Mundici were published. In the beginning of twenty-first century, Professor Wang had proposed theories of generalized tautologies, the algorithm of total implication of fuzzy reasoning and Quantitative logic that was published by Information Sciences in 2009. At present, these theories have been successfully applied to fuzzy controlling, fuzzy recognition, fuzzy systems and some propositional logic systems by Professor Li, Professor Liu, and other scholars. In Quantitative logic, the concept of truth degree of formulas in propositional logic systems takes an important and crucial role. In the present paper, we have introduced the concept of generalized truth degree to finitely-valued propositional logic system NMŁ which was proposed by Professor Wang and published by Fuzzy Sets and Systems in 2005. Because of the specific characteristic of definition of NMŁ logic algebra, in this paper, we introduce the concept of generalized truth degree only in $(3n+1)$-valued propositional logic system NMŁ. The work of this paper is a foundation for establishing the theory of quantitative logic in finitely-valued propositional logic system NMŁ.

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